TUTORIAL NOTES FOR MATH4220

JUNHAO ZHANG

1. DECAY PROPERTIES FOR WAVE EQUATIONS

Let us recall the solution to the Cauchy problems of wave equations.

Theorem 1. *Let* $u_0 \in C_c^3(\mathbb{R}^n)$*,* $u_1 \in C_c^2(\mathbb{R}^n)$*. Then*

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$
$$(u, \partial_t u)(0,x) = (u_0, u_1)(x), \quad x \in \mathbb{R}^n,$$

has a solution $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^n)$ *which is defined by*

$$u(t,x) = \begin{cases} \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy, & n=1, \\ \frac{1}{2\pi t^2}\int_{B_t(x)} \frac{tu_0(y) + t\nabla u_0(y)\cdot(y-x) + t^2 u_1(y)}{\sqrt{t^2 - |y-x|^2}} dy, & n=2, \\ \frac{1}{4\pi t^2}\int_{\partial B_t(x)} u_0(y) + \nabla u_0(y)\cdot(y-x) + tu_1(y) dS_y, & n=3. \end{cases}$$

In the following, we discuss the decay properties of the solutions to heat equation and wave equations.

Example 2. Let $n \in \{1,2,3\}$ and $u_1 \in C_c^\infty(\mathbb{R}^n)$. Then there exists a constant $C > 0$ which is independent of t such that for $t > 1$ the solution to

$$\partial_t^2 u - \Delta u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$
$$(u, \partial_t u)(0,x) = (0, u_1)(x), \quad x \in \mathbb{R}^n,$$

satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-\frac{n-1}{2}} \sum_{k=0}^{[\frac{n}{2}]} \|\nabla^k u_1\|_{L^1(\mathbb{R}^n)}.$$

Solution. For $n=1$, by the d'Alembert formula,

$$\|u(t)\|_{L^\infty(\mathbb{R})} \le C\|u_1\|_{L^1(\mathbb{R})}.$$

For $n=2$, the solution is

$$u(t,x) = \frac{1}{2\pi}\int_0^t \frac{r}{\sqrt{t^2-r^2}} \int_{|\omega|=1} u_1(x+r\omega) dS_\omega dr.$$

Then for $t > 1$, we write

$$\begin{aligned} u(t,x) =& \frac{1}{2\pi}\left(\int_0^{t-\frac{1}{2}} + \int_{t-\frac{1}{2}}^t\right) \frac{r}{\sqrt{t^2-r^2}} \int_{|\omega|=1} u_1(x+r\omega) dS_\omega dr \\ :=& \frac{1}{2\pi}(I_1 + I_2). \end{aligned}$$

Since

$$\begin{aligned}|I_1| \leq & \frac{1}{\sqrt{t^2-(t-\frac{1}{2})^2}} \int_0^{t-\frac{1}{2}} r \int_{|\omega|=1} |u_1(x+r\omega)| dS_\omega dr \\ \leq & \frac{1}{\sqrt{t-\frac{1}{4}}} \|u_1\|_{L^1(\mathbb{R}^2)},\end{aligned}$$

and

$$\begin{aligned}|I_2| \leq & \frac{1}{2\pi} \int_{t-\frac{1}{2}}^{t} \frac{r}{\sqrt{t^2-r^2}} \int_{|\omega|=1} \int_r^\infty |\partial_s u_1(x+s\omega)| ds dS_\omega dr \\ \leq & \|\nabla u_1\|_{L^1(\mathbb{R}^2)} \int_{t-\frac{1}{2}}^{t} \frac{1}{\sqrt{t^2-r^2}} dr \\ \leq & \sqrt{\frac{2}{t}} \|\nabla u_1\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Therefore

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\sqrt{1+t}} (\|u_1\|_{L^1(\mathbb{R}^2)} + \|\nabla u_1\|_{L^1(\mathbb{R}^2)}).$$

For $n=3$, by the Kirchhoff's formula,

$$u(t,x) = \frac{t}{4\pi} \int_{|\omega|=1} u_1(x+t\omega) dS_\omega,$$

then

$$\begin{aligned}|u(t,x)| \leq & \frac{t}{4\pi} \int_{|\omega|=1} \int_t^\infty |\partial_s u_1(x+s\omega)| ds dS_\omega \\ \leq & \frac{1}{4\pi t} \int_t^\infty s^2 \int_{|\omega|=1} |\nabla u_1(x+s\omega)| dS_\omega ds \\ \leq & \frac{1}{4\pi t} \|\nabla u_1\|_{L^1(\mathbb{R}^3)},\end{aligned}$$

for all $t > 0$.

A Supplementary Problem

Problem 3. If $u \equiv \partial_t u \equiv 0$ on $\{t=0\} \times B_{t_0}(x_0)$, then show that the solution to wave equation $\partial_t^2 u - \Delta u = 0$ satisfies that $u \equiv 0$ within $\{(t,x) : 0 \leq t \leq t_0, |x-x_0| \leq t_0 - t\}$.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

Email address: jhzhang@math.cuhk.edu.hk